

Group theoretical interpretation of the modified gravity in de Sitter space

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Abstract

A framework has been presented for theoretical interpretation of various modified gravitational models which is based on the group theoretical approach and unitary irreducible representations (UIR's) of de Sitter (dS) group. In order to illustrate the application of the proposed method, a model of modified gravity has been investigated. The background field method has been utilized and the linearized modified gravitational field equation has been obtained in the 4-dimensional dS space-time as the background. The field equation has been written as the eigen-value equation of the Casimir operators of dS space using the flat 5-dimensional ambient space notations. The Minkowskian correspondence of the theory has been obtained by taking the zero curvature limit. It has been shown that under some simple conditions, the linearized modified field equation transforms according to two of the UIR's of dS group labeled by $\Pi_{2,1}^{\pm}$ and $\Pi_{2,2}^{\pm}$ in the discrete series. It means that the proposed modified gravitational theory can be a suitable one to describe the quantum gravitational effects in its linear approximation on dS space. The field equation has been solved and the solution has been written as the multiplication of a symmetric rank-2 polarization tensor and a massless scalar field using the ambient space notations. Also the two-point function has been calculated in the ambient space formalism. It is dS invariant and free of any theoretical problems.

Keywords: Modified theories of gravity; Classical theories of gravity; Models of quantum gravity, de Sitter group, Linear gravity, de Sitter space-time.

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1 Introduction

It is well known that there are many good reasons to consider the Einstein general relativity as the best theory for the gravitational interaction, but according to the recent cosmological observations it seems that this theory may be incomplete. In addition to the well known problems of the Einstein general relativity in explaining the astrophysical phenomenology (i.e., the galactic rotation curves and small scale structure formation), recent cosmological data indicate an underlying cosmic acceleration of the universe which cannot be recast in the framework of the Einstein general relativity.

It is for these reasons and some other issues such as cosmic microwave background anisotropies [1], large scale structure formation [2], baryon oscillations [3] and weak lensing [4] that in recent years many authors are interested to generalize standard Einstein gravity. Among alternative proposed models the so-called extended theory of gravitation and, in particular, the gravity theories stem from nonlinear actions or higher-order theories of gravity have provided interesting results [5, 6, 7, 8, 9, 10]. These models are based on gravitational actions which are non-linear in the Ricci curvature \mathcal{R} and/ or contain terms involving combinations of derivatives of \mathcal{R} [11, 12, 13].

Recent astronomical observations of supernova and cosmic microwave background [14] indicate that the universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the universe accelerates indefinitely, the standard cosmology leads to an asymptotic dS universe. In addition, dS space-time plays an important role in the inflationary scenario where an exponentially expanding approximately dS space-time is employed to solve a number of problems in standard cosmology. Furthermore, the quantum field theory on dS space-time is also of considerable interest.

Furthermore, the gravitational field in the linear approximation behaves like a massless spin-2 particle which propagates on the background space-time. Following the Wigner's theorem, a linear gravitational field should transform according to the UIR's of the symmetric group of the background space-time. In this paper, dS space-time has been considered as the background. It has been shown that the proposed generalized Einstein's theory, in its linear approximation, can be associated with the UIR's of dS group.

The main goal of this work is to propose a theoretical framework for validity interpretation of the modified gravity theories, from group theoretical point of view, in dS space. The idea is that if a proposed model of modified gravity corresponds to the UIR's of dS group it can be considered as a possible successful model.

The organization of this paper is based on the following order. In section-2, a generalized Einstein-Hilbert gravitational action has been introduced and corresponding linear generalized Einstein gravitational field equation has been obtained in terms of the intrinsic dS coordinates as the background. Details of derivations have been given in appendices. Next, the linearized field equation has been written in terms of the Casimir operators of dS group making use of the five-dimensional ambient space formalism. The physical sector of the theory has been obtained by imposing the divergenceless and traceless conditions and the possible relations between this field equation and the UIR's of dS group have been investigated. By imposing a simple condition the conformally invariant theory of gravity is reproduced. In section-3, we obtained the solution to the conformally invariant field equation, using the ambient space notations. The solution can be written as the multiplication of a symmetric generalized polarization rank-2 tensor and

a massless minimally coupled scalar field in dS space. In section-4, we have calculated the conformally invariant two-point function, in terms of the massless minimally coupled scalar two-point function, using the ambient space formalism. It is dS invariant, symmetric and satisfies the traceless and divergenceless conditions. The results are summarized and discussed in section-5. Some useful mathematical relations and details of derivations of equations have been given in the appendices.

2 The field equation

The terms containing fourth order derivatives of the metric may be constructed out by curvature invariants (other than the cosmological constant), that is

$$\mathcal{R}, \quad \mathcal{R}^2, \quad \mathcal{R}_{ab}\mathcal{R}^{ab}, \quad \mathcal{R}_{abcd}\mathcal{R}^{abcd}.$$

Therefore, the gravitational action for the modified field equation in the 4-dimensional dS space-time with the metric signature $(-, +, +, +)$ can be written in the following general form

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[a_0(\mathcal{R} - 2\Lambda) + a_1\mathcal{R}^2 + a_2\mathcal{R}^{ab}\mathcal{R}_{ab} + a_3\mathcal{R}^{abcd}\mathcal{R}_{abcd} \right],$$

where $\Lambda = 3H^2$ is the positive cosmological constant. \mathcal{R}_{abcd} is the Riemann tensor, \mathcal{R}_{ab} is the Ricci tensor and $\mathcal{R} = g^{ab}\mathcal{R}_{ab}$ is the Ricci scalar of the space-time under consideration. a_0, a_1, a_2 and a_3 are constant coefficients. The coefficients a_1, a_2 and a_3 are positive with the dimension of $(\text{Length})^2$.

Taking note the fact that the Gauss-Bonnet action

$$\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(\mathcal{R}^2 - 4\mathcal{R}^{ab}\mathcal{R}_{ab} + \mathcal{R}^{abcd}\mathcal{R}_{abcd} \right)$$

is a total divergence. Adding it to the action will not contribute to the field equations and enable us to simplify the action somewhat and rewrite it as

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[a_0(\mathcal{R} - 2\Lambda) + a\mathcal{R}^2 + b\mathcal{R}^{ab}\mathcal{R}_{ab} \right], \quad (2.1)$$

with new coefficients. Therefore, including an $\mathcal{R}^{abcd}\mathcal{R}_{abcd}$ term is equivalent to altering the coefficients. The theory described by this action is referred to as fourth-order gravity, since it leads to fourth order equations. Numerous papers have been devoted to the study of fourth-order gravity.

Varying the action (2.1) with respect to the metric tensor g_{ab} the modified gravitational field equation is obtained as (Appendix-B)

$$a_0\mathcal{H}_{ab}^{(0)} + a\mathcal{H}_{ab}^{(1)} + b\mathcal{H}_{ab}^{(2)} = 0, \quad (2.2)$$

where $\mathcal{H}_{ab}^{(0)} = G_{ab} + \Lambda g_{ab}$ and $G_{ab} = \mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}g_{ab}$ is the Einstein tensor and

$$\mathcal{H}_{ab}^{(1)} = 2\mathcal{R}\mathcal{R}_{ab} - 2\nabla_a\nabla_b\mathcal{R} - \frac{1}{2}g_{ab}(\mathcal{R}^2 - 4\Box\mathcal{R}), \quad (2.3)$$

$$\mathcal{H}_{ab}^{(2)} = \square \mathcal{R}_{ab} - \nabla_c \nabla_a \mathcal{R}_b^c - \nabla_c \nabla_b \mathcal{R}_a^c + 2\mathcal{R}_a^c \mathcal{R}_{cb} - \frac{1}{2}g_{ab}(\mathcal{R}^{cd}\mathcal{R}_{cd} - 2\nabla_c \nabla_d \mathcal{R}^{cd}). \quad (2.4)$$

Making use of the relations $[\nabla_c, \nabla_a]\mathcal{R}_b^c = \mathcal{R}_{da}\mathcal{R}_b^d - \mathcal{R}_{bca}^d \mathcal{R}_d^c$, $\nabla_c \mathcal{R}_b^c = 1/2\nabla_b \mathcal{R}$, $\nabla_c \nabla_d \mathcal{R}^{cd} = 1/2\square \mathcal{R}$ and other symmetry properties of the Riemann tensor [10], it is easy to show that the field equation (2.2) is agree with Eq.(2.3) of ref. [15] with $\gamma = 0$.

2.1 Linear field equation in dS space

In order to obtain the linearized form of the field equation (2.2), one can use the background field method. That is $g_{ab} = g_{ab}^{(BG)} + h_{ab}$, in which $g_{ab}^{(BG)}$ is the background metric and h_{ab} is its fluctuations. Indices are raised and lowered by the background metric. We suppose that $g_{ab}^{(BG)} = g_{ab}^{(ds)} \equiv \tilde{g}_{ab}$. So one can write

$$g_{ab} = \tilde{g}_{ab} + h_{ab} \quad \text{and} \quad g^{ab} = \tilde{g}^{ab} - h^{ab}. \quad (2.5)$$

The metric \tilde{g}_{ab} is a solution to Einstein's field equation with the positive cosmological constant $\Lambda = 3H^2$:

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + 3H^2\tilde{g}_{ab} = 0. \quad (2.6)$$

Using the approximations given in Eq.(2.5), in Eq.(2.3), we have (Appendix-C)

$$\mathcal{H}_{ab}^{(0)} = \tilde{H}_{ab}^{(0)} + H_{ab}^{(0)}, \quad (2.7)$$

where $\tilde{H}_{ab}^{(0)}$ is the dS correspondent to $\mathcal{H}_{ab}^{(0)}$ and

$$H_{ab}^{(0)} = \frac{1}{2}(\nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac} - \square h_{ab} - \nabla_a \nabla_b h' + 2H^2 h_{ab}) + \frac{1}{2}\tilde{g}_{ab}(\square h' - \nabla_c \nabla_d h^{cd} + H^2 h'), \quad (2.8)$$

in which $h' = h_a^a$ is the trace of h_{ab} with respect to the background metric and ∇^b is the background covariant derivative. It is easy to show that (Appendix-D)

$$\mathcal{H}_{ab}^{(1)} = \tilde{H}_{ab}^{(1)} + H_{ab}^{(1)} \quad (2.9)$$

where $\tilde{H}_{ab}^{(1)}$ is the correspondent to $\mathcal{H}_{ab}^{(1)}$ in dS space and

$$\begin{aligned} H_{ab}^{(1)} = & +12H^2(\nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac} - \square h_{ab}) - 2\nabla_a \nabla_b (\nabla_c \nabla_d h^{cd} - \square h' + 3H^2 h') + 24H^4 h_{ab} \\ & - 2\tilde{g}_{ab} (3H^2 \nabla_c \nabla_d h^{cd} + 3H^4 h' - \square \nabla_c \nabla_d h^{cd} + \square^2 h'). \end{aligned} \quad (2.10)$$

It is easy to show that (Appendix-E)

$$\mathcal{H}_{ab}^{(2)} = \tilde{H}_{ab}^{(2)} + H_{ab}^{(2)} \quad (2.11)$$

where $\tilde{H}_{ab}^{(2)}$ is the correspondent to $\mathcal{H}_{ab}^{(2)}$ in dS space and

$$H_{ab}^{(2)} = \frac{1}{2} \left[\square (\nabla_a \nabla_c h_b^c + \nabla_b \nabla_c h_a^c) - 2H^2 \square h_{ab} - \square^2 h_{ab} + \nabla_a \nabla_b \square h' \right]$$

$$\begin{aligned}
& +2H^2 (\nabla_a \nabla_c h_b^c + \nabla_b \nabla_c h_a^c) - \nabla_a \nabla_b \nabla_c \nabla_d h^{cd} - 3H^2 \nabla_a \nabla_b h' + 4H^4 h_{ab} \\
& + \frac{1}{2} \tilde{g}_{ab} \left(2H^2 \nabla_c \nabla_d h^{cd} - 2H^4 h' + 7H^2 \square h' + \square \nabla_c \nabla_d h^{cd} - \square^2 h' \right). \tag{2.12}
\end{aligned}$$

Substituting Eqs.(2.8), (2.10) and (2.12) in Eq.(2.2), we have

$$a_0 H_{ab}^{(0)} + a H_{ab}^{(1)} + b H_{ab}^{(2)} = 0. \tag{2.13}$$

Eq.(2.13) is the linearized modified gravitational field equation in dS background, which has been written in terms of the intrinsic coordinates X_a of the 4-dimensional dS space-time. The linear field equation (2.13) can be written in the following explicit form

$$\begin{aligned}
& -\frac{b}{2} \square^2 h_{ab} - \left(\frac{a_0}{2} + 12aH^2 + bH^2 \right) \square h_{ab} + H^2 (a_0 + 24aH^2 + 4bH^2) h_{ab} \\
& + \left(2a + \frac{b}{2} \right) \nabla_a \nabla_b \square h' - \left(\frac{a_0}{2} + 6aH^2 + 3bH^2 \right) \nabla_a \nabla_b h' - (2a + b) \nabla_a \nabla_b \nabla_c \nabla_d h^{cd} \\
& + \left(\frac{a_0}{2} + 12aH^2 + 2bH^2 + \frac{b}{2} \square \right) (\nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac}) \\
& + \frac{1}{2} \tilde{g}_{ab} \left[(-a_0 - 12aH^2 - 2bH^2) \nabla_c \nabla_d h^{cd} + (a_0 - bH^2) \square h' \right. \\
& \left. + (a_0 - 12aH^2 - 2bH^2) H^2 h' + (4a + b) (\square \nabla_c \nabla_d h^{cd} - \square^2 h') \right] = 0. \tag{2.14}
\end{aligned}$$

The Minkowskian correspondence of the theory can be obtained by taking the zero curvature (i.e. $H \rightarrow 0$) of Eq.(2.14), it is

$$\begin{aligned}
& -\frac{1}{2} \left[b \square^2 h_{ab} + a_0 \square h_{ab} - (4a + b) \partial_a \partial_b \square h' - (a_0 + b \square) (\partial_a \partial^c h_{bc} + \partial_b \partial^c h_{ac}) + a_0 \partial_a \partial_b h' \right] \\
& + \frac{1}{2} \eta_{ab} \left[a_0 (\square h' - \partial_c \partial_d h^{cd}) + (4a + b) (\square \partial_c \partial_d h^{cd} - \square^2 h') \right] - (2a + b) \partial_a \partial_b \partial_c \partial_d h^{cd} = 0, \tag{2.15}
\end{aligned}$$

where η_{ab} is the metric and $\square = \eta_{ab} \partial^a \partial^b = \partial^a \partial_a$ is the wave operator in the flat space.

In order to obtain the physical sector of the model, one must to impose the physical conditions $\nabla_a h^{ab} = 0 = \nabla_b h^{ab}$ and $h' = 0$. In this case following Takook et al [16] we obtain

$$\left[-\frac{b}{2} \square^2 - \left(\frac{a_0}{2} + 12aH^2 + bH^2 \right) \square + H^2 (a_0 + 24aH^2 + 4bH^2) \right] h_{ab} = 0, \tag{2.16}$$

for the metric signature $(-, +, +, +)$, and

$$\left[-\frac{b}{2} \square^2 + \left(\frac{a_0}{2} + 12aH^2 + bH^2 \right) \square + H^2 (a_0 + 24aH^2 + 4bH^2) \right] h_{ab} = 0, \tag{2.17}$$

for the metric signature $(+, -, -, -)$.

In the following subsection, in order to consider the possible relations between the field equation and the UIR's of the dS group, the linearized field equation (2.17) will be written in terms of the Casimir operators of dS group, using the 5-dimensional ambient space notations.

2.2 dS group and Casimir operators in the field equation

The dS space-time is a maximally symmetric space-time having a positive constant curvature. It can be easily represented as a four-dimensional hyperboloid

$$\eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2}, \quad \alpha, \beta, \dots = 0, 1, 2, 3, 4, \quad (2.18)$$

embedded in a flat five-dimensional space with metric $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$. The dS metrics is

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta|_{x^2=-H^{-2}} = \tilde{g}_{ab}dX^a dX^b, \quad a, b, \dots = 0, 1, 2, 3, \quad (2.19)$$

where X^a 's are the 4 space-time intrinsic coordinates in dS hyperboloid. Different coordinate systems can be chosen [17]. Any geometrical object in this space can be written in terms of the four local intrinsic coordinates X^a or in terms of the five global ambient space coordinates x^α .

In order to express Eq.(2.17) in terms of the ambient space notations, originally developed by Christian Fronsdal [18], we adopt the tensor field $\mathcal{K}_{\alpha\beta}(x)$ in ambient space notations. Note that the ‘‘intrinsic’’ field $h_{ab}(X)$ is locally determined by the transverse tensor field $\mathcal{K}_{\alpha\beta}(x)$ through

$$h_{ab}(X) = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x^\beta}{\partial X^b} \mathcal{K}_{\alpha\beta}(x(X)). \quad (2.20)$$

In these notations, the solutions to the field equations are easily written out in terms of scalar fields. The reader how is not familiar to the ambient space notations is referred to [25] and references therein. The symmetric tensor field $\mathcal{K}_{\alpha\beta}(x)$ is defined on dS space-time and satisfies the transversality condition [19, 20]

$$x \cdot \mathcal{K}(x) = 0, \quad i.e. \quad x^\alpha \mathcal{K}_{\alpha\beta}(x) = 0, \quad \text{and} \quad x^\beta \mathcal{K}_{\alpha\beta}(x) = 0. \quad (2.21)$$

The covariant derivative in the ambient space notations is

$$D_\beta T_{\alpha_1 \dots \alpha_i \dots \alpha_n} = \bar{\partial}_\beta T_{\alpha_1 \dots \alpha_i \dots \alpha_n} - H^2 \sum_{i=1}^n x_{\alpha_i} T_{\alpha_1 \dots \beta \dots \alpha_n}, \quad (2.22)$$

where $\bar{\partial}$ is tangential (or transverse) derivative in dS space

$$\bar{\partial}_\alpha = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha x \cdot \partial, \quad x \cdot \bar{\partial} = 0, \quad (2.23)$$

$\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is the transverse projector. It is easily shown that the metric \tilde{g}_{ab} corresponds to the transverse projector $\theta_{\alpha\beta}$ that is

$$\tilde{g}_{ab}(X) = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x^\beta}{\partial X^b} \theta_{\alpha\beta}(x). \quad (2.24)$$

The kinematical group of dS space is the 10-parameter group $\text{SO}_0(1, 4)$ which is one of the two possible deformations of the Poincaré group. There are two Casimir operators

$$Q_s^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta}, \quad Q_s^{(2)} = -W_\alpha W^\alpha, \quad (2.25)$$

where

$$W_\alpha = -\frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}, \quad \text{with 10 infinitesimal generators} \quad L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}. \quad (2.26)$$

The subscript s in $Q_s^{(1)}$, $Q_s^{(2)}$ reminds that the carrier space is constituted by tensors of rank s . The orbital part $M_{\alpha\beta}$, and the action of the spinorial part $S_{\alpha\beta}$ on a rank-2 tensor field \mathcal{K} defined on the ambient space read respectively [20]

$$M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha), \quad S_{\alpha\beta} \mathcal{K}_{\gamma\delta} = -i(\eta_{\alpha\gamma} \mathcal{K}_{\beta\delta} - \eta_{\beta\gamma} \mathcal{K}_{\alpha\delta} + \eta_{\alpha\delta} \mathcal{K}_{\beta\gamma} - \eta_{\beta\delta} \mathcal{K}_{\alpha\gamma}). \quad (2.27)$$

The symbol $\epsilon_{\alpha\beta\gamma\delta\eta}$ holds for the usual antisymmetrical tensor. The action of the Casimir operator $Q_2^{(1)}$ on \mathcal{K} can be written in the more explicit form

$$Q_2^{(1)} \mathcal{K}(x) = (Q_0^{(1)} - 6) \mathcal{K}(x) + 2\eta \mathcal{K}' + 2\mathcal{S} x \partial \cdot \mathcal{K}(x) - 2\mathcal{S} \partial x \cdot \mathcal{K}(x), \quad (2.28)$$

where, $Q_0^{(1)} = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = -H^{-2} (\bar{\partial})^2$ is the scalar Casimir operator. The symmetrizer \mathcal{S} is defined for two vectors ξ_α and ω_β by $\mathcal{S}(\xi_\alpha \omega_\beta) = \xi_\alpha \omega_\beta + \xi_\beta \omega_\alpha$. \mathcal{K}' is the trace of the tensor \mathcal{K} and the action of the Casimir operator $Q_1^{(1)}$ on the vector K can be written in the more explicit form

$$Q_1^{(1)} K(x) = (Q_0^{(1)} - 2) K(x) + 2x \bar{\partial} \cdot K(x) + 2H^2 x x \cdot K(x) - 2\bar{\partial} x \cdot K(x). \quad (2.29)$$

As shown by Dixmier[21], the UIR,s of dS group have a classification scheme in terms of a pair of parameters (p, q) . The Casimir operators take the following possible spectral values:

$$\langle Q_p^{(1)} \rangle = -p(p+1) - (q+1)(q-2), \quad \langle Q_p^{(2)} \rangle = -p(p+1)q(q-1). \quad (2.30)$$

Depending on the different values of the pair of parameters (p, q) , three different series of representations are distinguishable: the principal, the complementary and the discrete series [21, 22]. Mathematical details of the group contraction and the physical principles underlying the relationship between dS and Poincaré groups can be found in Refs [23] and [24] respectively. The spin-2 tensor representations relevant to the present work are [25]:

- i) The UIR's of the principal series labeled by $U^{2,\nu}$ with $p = s = 2$ and $q = \frac{1}{2} + i\nu$ correspond to the Casimir spectral values:

$$\langle Q_2^{(1)} \rangle = \nu^2 - \frac{15}{4}, \quad \nu \in \mathbb{R}, \quad (2.31)$$

note that $U^{2,\nu}$ and $U^{2,-\nu}$ are equivalent.

- ii) The UIR's of the complementary series denoted by $V^{2,q}$ with $p = s = 2$ and $q - q^2 = \mu$, correspond to the following spectral values

$$\langle Q_2^{(1)} \rangle = q - q^2 - 4 \equiv \mu - 4, \quad 0 < \mu < \frac{1}{4}. \quad (2.32)$$

- iii) The UIR's of the discrete series conventionally labeled by $\Pi_{2,q}^\pm$ in which $p = s = 2$ and takes the following spectral values

$$\langle Q_2^{(1)} \rangle = -6 - (q+1)(q-2), \quad q = 1, 2. \quad (2.33)$$

The “massless” spin-2 field in dS space corresponds to the $\Pi_{2,2}^{\pm}$ and $\Pi_{2,1}^{\pm}$ cases in which the sign \pm , stands for the helicity. In these cases, the two representations $\Pi_{2,2}^{\pm}$, in the discrete series with $p = q = 2$, have a Minkowskian interpretation. It is important to note that the representations $\Pi_{2,1}^{\pm}$ do not have corresponding flat limit [25]. (More details can be found in [20] and references therein.)

We now attempt to express the wave equation (2.17) in terms of the Casimir operators of dS group. The d'Alembertian operator becomes [26]

$$\square h_{ab} = \nabla^c \nabla_c h_{ab} = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x^\beta}{\partial X^b} \left[-H^2 Q_0^{(1)} - 2H^2 \right] \mathcal{K}_{\alpha\beta}, \quad (2.34)$$

and

$$\square^2 h_{ab} = \nabla^c \nabla_c \nabla^d \nabla_d h_{ab} = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x^\beta}{\partial X^b} \left[H^4 \left(Q_0^{(1)} \right)^2 + 4H^4 Q_0^{(1)} + 4H^4 \right] \mathcal{K}_{\alpha\beta}, \quad (2.35)$$

where, the conditions of tracelessness and divergence free (e.i. $\bar{\partial} \cdot \mathcal{K} = 0 = \mathcal{K}'$), have been imposed to the physical states. By use of the above equations in Eq. (2.17) we have

$$\left[\frac{b}{2} H^2 \left(Q_0^{(1)} \right)^2 + \left(\frac{a_0}{2} + 12aH^2 + 3bH^2 \right) Q_0^{(1)} \right] \mathcal{K}_{\alpha\beta} = 0. \quad (2.36)$$

In terms of different choice of coefficients in the proposed action (2.1) different gravitational theories may be achieved. Now the following various choices are considerable

- By choosing $a = 0$ and $b = 0$ we return to the physical linear pure dS theory, that is

$$Q_0^{(1)} \mathcal{K}_{\alpha\beta} = 0, \quad \text{or} \quad \left(Q_2^{(1)} + 6 \right) \mathcal{K}_{\alpha\beta} = 0. \quad (2.37)$$

This is an eigen-value equation with the eigen-value $\langle Q_2^{(1)} \rangle = -6$. From the group theoretical point of view this corresponds to UIR's of dS group labeled by $\Pi_{2,2}^{\pm}$ in the discrete series which reduces to the physical representations of the Poincaré group in the zero curvature limit. This is why it is called as the physical state. It has been discussed in [25], for the gauge-fixed value equal to zero, [27] for the gauge-fixed value equal to $\frac{2}{5}$ and the extended discussions are given in [28].

• Letting $a_0 = 1$, $b = 0$, the model reduces to a $f(\mathcal{R})$ theory model with $f(\mathcal{R}) = \mathcal{R} + a\mathcal{R}^2$. It is known as a relatively successful model, which explains the inflation and positive acceleration of the universe [29, 30]. Under these conditions, the linearized field equation (2.39) reduces to

$$(1 + 24aH^2) Q_0^{(1)} \mathcal{K}_{\alpha\beta} = 0, \quad \text{or} \quad (1 + 24aH^2) \left(Q_2^{(1)} + 6 \right) \mathcal{K}_{\alpha\beta} = 0. \quad (2.38)$$

It corresponds to the UIR's of dS group labeled by $\Pi_{2,2}^{\pm}$ in the discrete series too. This is why the model is a successful one. The field equation (2.38) has been considered in ref. [31].

• One may set $a_0 = 0$, $a = -\frac{1}{3}$ and $b = 1$, by which the theory reduces to the Weyl conformal theory with the linearized field equation

$$Q_0^{(1)} \left(Q_0^{(1)} - 2 \right) \mathcal{K}_{\alpha\beta} = 0, \quad \text{or} \quad \left(Q_2^{(1)} + 6 \right) \left(Q_2^{(1)} + 4 \right) \mathcal{K}_{\alpha\beta} = 0. \quad (2.39)$$

The same equation has been obtained by Dehghani, et. al from a different approach in [25].

The field equation $(Q_2^{(1)} + 4) \mathcal{K}_{\alpha\beta} = 0$, is also an eigen-value equation with the eigen-value $\langle Q_2^{(1)} \rangle = -4$. It corresponds to one of the UIR's of dS group denoted by $\Pi_{2,1}^\pm$ in the discrete series with the same Poincaré correspondence as $\Pi_{2,2}^\pm$ in the zero curvature limit. Indeed two of UIR's of dS group have only one Poincaré correspondence. It has been discussed in [32].

As it is clear with the help of above-mentioned examples, we believe that it is necessary for any successful theory of gravity to transform according to the UIR,s of dS group. In other words if a model of modified gravity theory does not correspond to the UIR,s of dS group in its linear approximations it can not produce valid and helpful physical results.

For the general discussion on the proposed modified gravity theory, let $A = aH^2$ and $B = bH^2$. In terms of these dimensionless coefficients the field equation (2.36) can be written as

$$\left[(Q_0^{(1)})^2 + \left(\frac{a_0}{B} + 24\frac{A}{B} + 6 \right) Q_0^{(1)} \right] \mathcal{K}_{\alpha\beta} = 0, \quad B \neq 0. \quad (2.40)$$

As a direct mathematical result, the proposed model in it's linear approximation, generally transforms according to the UIR's of dS group and it is a suitable candidate model of gravitation on dS space if the characteristic equation

$$\frac{a_0}{B} + 24\frac{A}{B} + 8 = 0, \quad B \neq 0, \quad (2.41)$$

is satisfied. Under this condition it describes a massless spin-2 particle (the graviton, if it exists) in it's linear approximation and transforms according to two of UIR,s of dS group. We therefore believe that it can be a successful modified gravity theory. For more clarity, in the following sections, we solve the field equation (2.40), with the condition (2.41), using the ambient space formalism. Also we obtain the two-point function for the linearized theory of gravitation making use of the ambient space notations, and show that the results are free of any theoretical problems.

3 Solution to the conformal field equation

A general solution of to the conformal field equation can be constructed from the combination of a scalar field and two vector fields. Let us first introduce a traceless and transverse tensor field \mathcal{K} in terms of a five-dimensional constant vector $Z_1 = (Z_{1\alpha})$ and a scalar field ϕ_1 and two vector fields K and K_g by putting [20, 25, 27, 28, 32, 33]

$$\mathcal{K} = \theta\phi_1 + \mathcal{S}\bar{Z}_1 K + D_2 K_g, \quad (3.1)$$

where D_2 is the generalized gradient operator defined by $D_2 K = \mathcal{S}(D_1 + x)K$, $D_{1\alpha} = H^{-2}\bar{\partial}_\alpha$ and $\bar{Z}_{1\alpha} = \theta_{\alpha\beta}Z_1^\beta$. Taking the trace of $\mathcal{K}_{\alpha\beta}$ we have

$$\mathcal{K}' = 4\phi_1 + 2Z_1.K + 2H^2(x.Z_1)x.K + 2D_1.K_g - 2x.K_g = 0, \quad (3.2)$$

Using the ansatz (3.1) to the field equation we have (Appendix F)

$$\left\{ \begin{array}{ll} (Q_0^{(1)} + 4)(Q_0^{(1)} + 6)\phi_1 + 8(Q_0^{(1)} + 2)Z_1.K = 0, & (a) \\ Q_1^{(1)}(Q_1^{(1)} + 2)K = 0, \quad \text{or} \quad Q_1^{(1)}Q_0^{(1)}K = 0, \quad \partial.K = 0 = x.K, & (b) \\ (Q_1^{(1)} + 4)(Q_1^{(1)} + 6)K_g = 4H^2[(Q_1^{(1)} + 5)x.Z_1K + Z_1.D_1K - xZ_1.K]. & (c) \end{array} \right. \quad (3.3)$$

The vector field K can be written in the following general form

$$K_\alpha = \bar{Z}_{2\alpha}\phi_2 + D_{1\alpha}\phi_3, \quad (3.4)$$

where Z_2 is another constant 5-vector and ϕ_2 and ϕ_3 are two arbitrary scalar fields, should be determined. Using the divergenceless condition we have

$$Q_0^{(1)}\phi_3 = Z_2.\bar{\partial}\phi_2 + 4H^2(x.Z_2)\phi_2, \quad (3.5)$$

and substituting Eq.(3.4) in Eq.(3.3-b) leads to the following two equations

$$Q_0^{(1)}(Q_0^{(1)} - 2)\phi_2 = 0, \quad (3.6)$$

$$Q_0^{(1)}(Q_0^{(1)} + 2)\phi_3 = 4H^2Q_0^{(1)}[(x.Z_2)\phi_2] + 8H^2(x.Z_2)\phi_2 + 4Z_2.\bar{\partial}\phi_2. \quad (3.7)$$

The Eq.(3.6) has a dS plane wave solution of the form

$$\phi_2 = (Hx.\xi)^\sigma, \quad \xi^2 = 0, \quad \text{with} \quad \sigma(\sigma + 3)(\sigma + 2)(\sigma + 1) = 0. \quad (3.8)$$

Note that ϕ_2 is the minimally coupled scalar field for $\sigma = 0, -3$. In that case it obeys the field equation $Q_0^{(1)}\phi_2 = 0$ [25, 34]. Also ϕ_2 is the conformally coupled scalar field for $\sigma = -1, -2$ and satisfies the field equation $(Q_0^{(1)} - 2)\phi_2 = 0$ [35].

Substituting $Q_0^{(1)}\phi_3$ and $(Q_0^{(1)})^2\phi_3$ from Eq.(3.5) into Eq.(3.7), we obtain

$$Q_0^{(1)}Z_2.\bar{\partial}\phi_2 = 2Z_2.\bar{\partial}\phi_2. \quad (3.9)$$

Now regarding Eqs.(3.6) and (3.9) and using the identity

$$Q_0^{(1)}[(x.Z_2)\phi_2] = (x.Z_2)Q_0^{(1)}\phi_2 - 4(x.Z_2)\phi_2 - 2Z_2.D_1\phi_2, \quad (3.10)$$

we obtain

$$Q_0^{(1)}[(x.Z_2)Q_0^{(1)}\phi_2] = 2Q_0^{(1)}[(x.Z_2)\phi_2] + 8(x.Z_2)\phi_2 + 4Z_2.D_1\phi_2. \quad (3.11)$$

Combining Eqs.(3.9) and (3.11) we have

$$(x.Z_2)\phi_2 = \frac{1}{8}Q_0^{(1)} \left[(x.Z_2)Q_0^{(1)}\phi_2 - 2(x.Z_2)\phi_2 - 2Z_2.D_1\phi_2 \right]. \quad (3.12)$$

Substituting Eqs.(3.9) and (3.12) in Eq.(3.5) we have

$$\phi_3 = \frac{1}{2} \left[H^2(x.Z_2)Q_0^{(1)}\phi_2 - Z_2.\bar{\partial}\phi_2 - 2H^2(x.Z_2)\phi_2 \right]. \quad (3.13)$$

Now the solution to Eq.(3.3-b) can be written in terms of the dS massless scalar field $\phi_2 \equiv \phi_s$ as

$$K_\alpha = \bar{Z}_{2\alpha}\phi_s + \frac{1}{2}D_{1\alpha} \left[H^2(x.Z_2)Q_0^{(1)} - Z_2.\bar{\partial} - 2H^2x.Z_2 \right] \phi_s. \quad (3.14)$$

The explicit form of the vector field K_α is

$$K = \frac{\sigma}{2} \left[(\sigma + 2)\bar{Z}_2 + (\sigma^2 + 2\sigma - 2)\frac{x.Z_2}{x.\xi}\bar{\xi} \right] \phi_s, \quad (3.15)$$

and the condition of $\bar{\partial}.K_\alpha = 0$ can be written in the following explicit form

$$\bar{\partial}.K = \frac{1}{2}\sigma^2(\sigma + 3)(\sigma + 4)(x.Z_1)(x.Z_2)\phi_s = 0. \quad (3.16)$$

Noting Eq.(3.8), it is valid only for $\sigma = 0, -3$. As pointed out before we can treat the scalar field ϕ_s as the massless minimally coupled scalar field. Furthermore under these circumstances the vector field K_α satisfies the relation

$$Q_0^{(1)}K_\alpha = 0, \quad \text{or} \quad (Q_1^{(1)} + 2)K = 0. \quad (3.17)$$

It is easy to show that Eq.(3.3-a) has a solution of the form

$$\phi_1 = -\frac{2}{3}Z_1.K, \quad Q_0^{(1)}(Q_0^{(1)} - 2)\phi_1 = 0. \quad (3.18)$$

It means that ϕ_1 satisfies the scalar massless field equation in dS space [34, 35]. Now Eq (3.2) can be written as

$$\bar{\partial}.K_g = \frac{1}{3}H^2Z_1.K, \quad x.K_g = 0. \quad (3.19)$$

Making use of the relation

$$(Q_1^{(1)} + 5)x.Z_1K = 2x(Z_1.K) - 2(Z_1.D_1)K - (x.Z_1)K,$$

Eq.(3.3-c) can be written as

$$(Q_1^{(1)} + 4)(Q_1^{(1)} + 6)K_g = 4H^2[xZ_1.K - x.Z_1K - Z_1.D_1K]. \quad (3.20)$$

Now using the identities

$$6(x.Z_1)K = (Q_1^{(1)} + 6) \left[(x.Z_1)K + \frac{1}{9}D_1(Z_1.K) \right],$$

$$2(xZ_1.K - Z_1.D_1K) = (Q_1^{(1)} + 6)(x.Z_1K),$$

in Eq.(3.20) we obtain

$$(Q_1^{(1)} + 4)K_g = \frac{4}{3}H^2 \left[(x.Z_1)K - \frac{1}{18}D_1(Z_1.K) \right]. \quad (3.21)$$

It is easy to show that

$$4D_1(Z_1.K) = (Q_1^{(1)} + 4)D_1(Z_1.K), \quad (3.22)$$

$$4(x.Z_1)K = (Q_1^{(1)} + 4) \left[(x.Z_1)K + \frac{1}{6}D_1(Z_1.K) \right]. \quad (3.23)$$

Combining Eqs.(3.21)-(3.23) results in

$$K_g = \frac{1}{3}H^2 \left[(x.Z_1)K + \frac{1}{9}D_1(Z_1.K) \right]. \quad (3.24)$$

It satisfies the conditions given in Eq.(3.19).

Substituting Eqs.(3.15), (3.18) and (3.24) in Eq.(3.1) one can show that

$$\mathcal{K}_{\alpha\beta}(x) = \mathcal{E}_{\alpha\beta}(x, \xi, Z_1, Z_2)\phi_s, \quad (3.25)$$

where ϕ_s is a massless scalar field in dS space and \mathcal{E} is a generalized symmetric polarization tensor,

$$\mathcal{E} = \frac{\sigma}{2} \left[-\frac{2}{3}\theta Z_1. + \mathcal{S}\bar{Z}_1 + H^2\frac{1}{3}D_2(x.Z_1 + \frac{1}{9}D_1Z_1.) \right] \left[(\sigma + 2)\bar{Z}_2 + (\sigma^2 + 2\sigma - 2)\frac{x.Z_2}{x.\xi}\bar{\xi} \right]. \quad (3.26)$$

It is consistent with the results in [25] and [28] with $c = 0$.

4 The conformal two-point function

The two-point function $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x, x')$, which is a solution of the wave equation with respect to x or x' , can be found simply in terms of the scalar two-point function. Very similar to the recurrence formula (3.1) let us try the following possibility [20, 25, 27, 28, 32, 33]

$$\mathcal{W}(x, x') = \theta\theta'\mathcal{W}_0(x, x') + \mathcal{S}\mathcal{S}'\theta.\theta'\mathcal{W}_1(x, x') + D_2D_2'\mathcal{W}_g(x, x'), \quad (4.1)$$

where \mathcal{W} , \mathcal{W}_1 and \mathcal{W}_g are transverse bi-vectors, \mathcal{W}_0 is bi-scalar and $D_2D_2' = D_2'D_2$. Substituting the two-point function (4.1) in the field equation with respect to x , we have

$$\begin{cases} (Q_0^{(1)} + 4)(Q_0^{(1)} + 6)\theta'\mathcal{W}_0 + 8(Q_0^{(1)} + 2)\mathcal{S}'\theta'.\mathcal{W}_1, & (a) \\ Q_1^{(1)}(Q_1^{(1)} + 2)\mathcal{W}_1 = 0, \quad \text{or} \quad Q_1^{(1)}Q_0^{(1)}\mathcal{W}_1 = 0, \quad \partial.\mathcal{W}_1 = 0, & (b) \\ (Q_1^{(1)} + 4)(Q_1^{(1)} + 6)D_2'\mathcal{W}_g = 4H^2\mathcal{S}' \left[(Q_1^{(1)} + 5)(x.\theta')\mathcal{W}_1 + \theta'.D_1\mathcal{W}_1 + x\theta'.\mathcal{W}_1 \right]. & (c) \end{cases} \quad (4.2)$$

The solution to Eq.(4.2-b) has the following general form

$$\mathcal{W}_1 = \theta.\theta'\mathcal{W}_2 + D_1D_1'\mathcal{W}_3, \quad \text{and} \quad D_1'\mathcal{W}_3 = \frac{1}{2} \left[H^2(x.\theta')Q_0^{(1)} - \theta'.\bar{\partial} - 2H^2x.\theta' \right] \mathcal{W}_2, \quad (4.3)$$

in which $\mathcal{W}_2 \equiv \mathcal{W}_s$ is the massless minimally coupled scalar two-point function. The dS-invariance two-point function for the massless minimally coupled scalar field in the ‘‘Gupta-Bleuler vacuum’’ state is [36]

$$\mathcal{W}_s(x, x') = \frac{iH^2}{8\pi^2} \epsilon(x^0 - x'^0) [\delta(1 - \mathcal{Z}(x, x')) + \vartheta(\mathcal{Z}(x, x') - 1)], \quad (4.4)$$

with

$$\mathcal{Z} = -H^2x.x', \quad \text{and} \quad \epsilon(x^0 - x'^0) = \begin{cases} 1 & x^0 > x'^0, \\ 0 & x^0 = x'^0, \\ -1 & x^0 < x'^0. \end{cases} \quad (4.5)$$

In summary, the solution to the above system of equations is

$$\mathcal{W}_1 = \left[\theta.\theta' + \frac{1}{2}D_1 \left(H^2x.\theta'Q_0^{(1)} - \theta'.\bar{\partial} - 2H^2x.\theta' \right) \right] \mathcal{W}_s, \quad (4.6)$$

$$\theta' \mathcal{W}_0(x, x') = -\frac{2}{3} \mathcal{S}' \theta' \cdot \mathcal{W}_1(x, x'), \quad (4.7)$$

$$D'_2 \mathcal{W}_g(x, x') = \frac{1}{3} H^2 \mathcal{S}' \left[(x \cdot \theta') \mathcal{W}_1 + \frac{1}{9} D_1(\theta' \cdot \mathcal{W}_1) \right]. \quad (4.8)$$

The two-point function (4.1) also satisfies the field equation with respect to x' , in this case one can obtain

$$\begin{cases} (Q_0'^{(1)} + 4)(Q_0'^{(1)} + 6) \theta \mathcal{W}_0 + 8(Q_0'^{(1)} + 2) \mathcal{S} \theta \cdot \mathcal{W}_1, & (a) \\ Q_1'^{(1)}(Q_1'^{(1)} + 2) \mathcal{W}_1 = 0, \quad \text{or} \quad Q_1'^{(1)} Q_0'^{(1)} \mathcal{W}_1 = 0, \quad \partial' \cdot \mathcal{W}_1 = 0, & (b) \\ (Q_1'^{(1)} + 4)(Q_1'^{(1)} + 6) D_2 \mathcal{W}_g = 4 H^2 \mathcal{S} \left[(Q_1'^{(1)} + 5)(x' \cdot \theta) \mathcal{W}_1 + \theta \cdot D'_1 \mathcal{W}_1 - x' \theta \cdot \mathcal{W}_1 \right]. & (c) \end{cases} \quad (4.9)$$

with the solutions

$$\mathcal{W}_1 = \left[\theta' \cdot \theta + \frac{1}{2} D'_1 \left(H^2 x' \cdot \theta Q_0'^{(1)} - \theta \cdot \bar{\partial}' - 2 H^2 x' \cdot \theta \right) \right] \mathcal{W}_s, \quad (4.10)$$

$$\theta \mathcal{W}_0(x, x') = -\frac{2}{3} \mathcal{S} \theta \cdot \mathcal{W}_1(x, x'), \quad (4.11)$$

$$D_2 \mathcal{W}_g(x, x') = \frac{1}{3} H^2 \mathcal{S} \left[(x' \cdot \theta) \mathcal{W}_1 + \frac{1}{9} D'_1(\theta \cdot \mathcal{W}_1) \right]. \quad (4.12)$$

Note that the primed operators act on the primed coordinates only.

Making use of Eqs.(4.6)-(4.8) or (4.10)-(4.12) one can show that the conformal two-point function can be written as

$$\mathcal{W}_{\alpha\beta\alpha'\beta'} = \Delta_{\alpha\beta\alpha'\beta'} \mathcal{W}_s, \quad (4.13)$$

where

$$\begin{aligned} \Delta = \frac{1}{6} & \left[-2\theta \mathcal{S}' \theta' \cdot + \mathcal{S} \mathcal{S}' \theta \cdot \theta' + H^2 D_2 \mathcal{S}'(x \cdot \theta' + \frac{1}{3} D_1 \theta' \cdot) \right] \\ & \times \left[2\theta \cdot \theta' + D_1 \left(H^2 x \cdot \theta' Q_0'^{(1)} - \theta' \cdot \bar{\partial} - 2 H^2 x \cdot \theta' \right) \right]. \end{aligned} \quad (4.14)$$

It agrees with the results in [25] and [28] with $c = 0$.

5 Conclusion

According to the recent cosmological observations it seems that the standard Einstein theory of gravity may be incomplete and many attempts have been made to modify this theory. The so-called modified theory of gravitation and, in particular, non-linear gravity theories or higher-order theories of gravity have provided interesting results. The proposed models are based on gravitational actions which are non-linear in the Ricci curvature and constructed out by curvature invariants.

This work is devoted to an extension of the Einstein-Hilbert gravitational action, which is constructed out by the linear combination of Ricci scalar and Ricci tensor invariants in dS space. Varying the proposed action with respect to metric tensor leads to a fourth order gravitational field equation, conventionally named as the modified gravitational theory. The background field

method is utilized and the linearized field equation is obtained in terms of intrinsic coordinates in the 4-dimensional dS space as the background.

The gravitational field in the linear approximation behaves like a massless spin-2 particle which propagates on the background space-time. According to Wigner's theorem, a linear gravitational field should transform according to the UIR's of the symmetry group of the background space-time. In order to investigate the possible relations between the field equation and the UIR's of dS group it is transformed into the flat five-dimensional ambient space and the linearized field equation is written in terms of the Casimir operators of dS group. We obtained the Minkowskian correspondence of the theory by taking the zero curvature limit. The physical sector of the theory is obtained by imposing the divergenceless and traceless conditions. Some interesting theories are reproduced as the special cases of the theory and their validity and successfulness are discussed from group theoretical point of view. We demonstrated that it is necessary for a theory to be successful, in dS space-time, if it transforms according to the UIR's of dS group. We showed that the proposed theory transforms according to the UIR's of dS group if the constant coefficients satisfy some simple conditions. As a result this theory can be used as a successful model for solving the problems in the framework of quantum gravity.

As an special case of the theory the linearized Weyl theory of gravity is reproduced which transforms according to two of the UIR's of dS group denoted by $\Pi_{2,2}^\pm$ and $\Pi_{2,1}^\pm$ in discrete series. We obtained the solution to the conformally invariant field equation, using the ambient space notations. The solution can be written as the multiplication of a symmetric rank-2 generalized polarization tensor and a massless minimally coupled scalar field in dS space. Also we have calculated the conformally invariant two-point function, in terms of the basic bi-vectors of the ambient space. It is dS invariant, symmetric and satisfies the traceless and divergenceless conditions. We therefore claim that the proposed modified gravity theory under the given restrictions is a successful one and the introduced procedure can be used as a theoretical testing for the validity and successfulness of any given modified theory of gravity.

A Some useful mathematical relations

The following relations have been used in deriving the linearized field equations.

$$\tilde{R}_{abcd} = H^2(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc}), \quad (\text{A.1})$$

$$\tilde{R}_{ab} = 3H^2\tilde{g}_{ab}, \quad (\text{A.2})$$

$$\tilde{R} = 12H^2, \quad (\text{A.3})$$

$$(\mathcal{R}^c{}_{dab})_L \equiv \delta R^c{}_{dab} = \frac{1}{2} [\nabla_a (\nabla_d h_b^c + \nabla_b h_d^c - \nabla^c h_{db}) - \nabla_b (\nabla_d h_a^c + \nabla_a h_d^c - \nabla^c h_{ad})]. \quad (\text{A.4})$$

$$(\mathcal{R}_{ab})_L \equiv \delta R_{ab} = \frac{1}{2} (\nabla_a \nabla_c h_b^c + \nabla_b \nabla_c h_a^c + 8H^2 h_{ab} - 2H^2 h' \tilde{g}_{ab} - \square h_{ab} - \nabla_a \nabla_b h'). \quad (\text{A.5})$$

$$(\mathcal{R})_L \equiv \delta R = \nabla_c \nabla_b h^{cb} - \square h' - 3H^2 h'. \quad (\text{A.6})$$

$$(\mathcal{R}_d^c)_L \equiv \delta R_d^c = \frac{1}{2} (\nabla^c \nabla_a h_d^a + \nabla_d \nabla_a h^{ac} + 8H^2 h_d^c - 2H^2 h' \tilde{g}_d^c - \square h_d^c - \nabla^c \nabla_d h') - 3H^2 h_d^c. \quad (\text{A.7})$$

$$(\mathcal{R}^{bc})_L \equiv \delta R^{bc} = \frac{1}{2} \left(\nabla^c \nabla_a h^{ab} + \nabla^b \nabla_a h^{ac} + 8H^2 h^{bc} - 2H^2 h' \tilde{g}^{bc} - \square h^{bc} - \nabla^c \nabla^b h' \right) - 6H^2 h^{bc}. \quad (\text{A.8})$$

$$(\nabla_a \nabla_b \mathcal{R})_L \equiv \delta \nabla_a \nabla_b R = \nabla_a \nabla_b \delta R = \nabla_a \nabla_b \left(\nabla_c \nabla_d h^{cd} - \square h' - 3H^2 h' \right). \quad (\text{A.9})$$

$$(\square \mathcal{R})_L \equiv \delta \square R = \square \delta R = \square \left(\nabla_c \nabla_d h^{cd} - \square h' - 3H^2 h' \right). \quad (\text{A.10})$$

$$(\nabla_a \nabla_b \mathcal{R}_{cd})_L \equiv \delta \nabla_a \nabla_b R_{cd} = \frac{1}{2} \nabla_a \nabla_b \left(\nabla_c \nabla_e h_d^e + \nabla_d \nabla_e h_c^e + 2H^2 h_{cd} - 2H^2 h' \tilde{g}_{cd} - \square h_{cd} - \nabla_c \nabla_d h' \right). \quad (\text{A.11})$$

$$(\nabla_a \nabla_b \mathcal{R}_d^c)_L \equiv \delta \nabla_a \nabla_b R_d^c = \frac{1}{2} \nabla_a \nabla_b \left(\nabla^c \nabla_e h_d^e + \nabla_d \nabla_e h^{ce} + 2H^2 h_d^c - 2H^2 h' \tilde{g}_d^c - \square h_d^c - \nabla^c \nabla_d h' \right), \quad (\text{A.12})$$

$$(\nabla_a \nabla_b \mathcal{R}^{cd})_L \equiv \delta \nabla_a \nabla_b R^{cd} = \frac{1}{2} \nabla_a \nabla_b \left(\nabla^c \nabla_e h^{ed} + \nabla^d \nabla_e h^{ce} + 2H^2 h^{cd} - 2H^2 h' \tilde{g}^{cd} - \square h^{cd} - \nabla^c \nabla^d h' \right). \quad (\text{A.13})$$

$$(\square \mathcal{R}_{cd})_L \equiv \delta \square R_{cd} = \frac{1}{2} \square \left(\nabla_c \nabla_e h_d^e + \nabla_d \nabla_e h_c^e + 2H^2 h_{cd} - 2H^2 h' \tilde{g}_{cd} - \square h_{cd} - \nabla_c \nabla_d h' \right). \quad (\text{A.14})$$

B Details of derivation of Eq.(2.2)

In this subsection we obtain the field equation through variation of the action. The action (2.1) can be written as

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[f(\mathcal{R}) + b \mathcal{R}^{ab} \mathcal{R}_{ab} \right], \quad f(\mathcal{R}) = a_0(\mathcal{R} - 2\Lambda) + a \mathcal{R}^2, \quad (\text{B.1})$$

$$\delta \left[\sqrt{-g} f(\mathcal{R}) \right] = f(\mathcal{R}) \delta \sqrt{-g} + \sqrt{-g} f'(\mathcal{R}) \delta \mathcal{R}, \quad (\text{B.2})$$

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab}, \quad (\text{B.3})$$

$$\delta \mathcal{R} = \delta \left(g^{ab} \mathcal{R}_{ab} \right) = \mathcal{R}_{ab} \delta g^{ab} + g^{ab} \delta \mathcal{R}_{ab}, \quad (\text{B.4})$$

$$\delta \mathcal{R}_{ab} = \nabla_c \delta \Gamma_{ab}^c - \nabla_b \delta \Gamma_{ac}^c. \quad (\text{B.5})$$

The statement in Eq.(B.5) is the difference of two connections, it transforms as a tensor. one can show that

$$\delta \Gamma_{ab}^c = \frac{1}{2} g^{cd} \left(\nabla_a \delta g_{bd} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ab} \right), \quad (\text{B.6})$$

and substituting in Eq.(B.5) we have

$$\delta \mathcal{R}_{ab} = \frac{1}{2} \left(\nabla^c \nabla_a \delta g_{bc} + \nabla^c \nabla_b \delta g_{ac} - \nabla_a \nabla_b g^{cd} \delta g_{cd} - \square \delta g_{ab} \right). \quad (\text{B.7})$$

Now return to Eq.(B.4) we have

$$\delta\mathcal{R} = \mathcal{R}_{ab}\delta g^{ab} + g_{ab}\square\delta g^{ab} - \nabla_a\nabla_b\delta g^{ab}. \quad (\text{B.8})$$

Using Eq.(B.3) and Eq.(B.8) in Eq.(B.2) we have

$$\delta\left[\sqrt{-g}f(\mathcal{R})\right] = \sqrt{-g}\left[f'(\mathcal{R})\mathcal{R}_{ab} - \frac{1}{2}g_{ab}f(\mathcal{R}) + f'(\mathcal{R})(g_{ab}\square - \nabla_a\nabla_b)\right]\delta g^{ab}. \quad (\text{B.9})$$

$$\delta\left(\mathcal{R}^{ab}\mathcal{R}_{ab}\right) = \delta\left(g^{ca}g^{bd}\mathcal{R}_{cd}\mathcal{R}_{ab}\right) = 2\left(\mathcal{R}_c^a\mathcal{R}_{ab}\delta g^{bc} + \mathcal{R}^{ab}\delta\mathcal{R}_{ab}\right), \quad (\text{B.10})$$

and noting Eq.(B.7) we can show

$$\delta\left(\mathcal{R}^{ab}\mathcal{R}_{ab}\right) = 2\mathcal{R}_c^a\mathcal{R}_{ab}\delta g^{bc} + \mathcal{R}^{ab}\left(\nabla^c\nabla_a\delta g_{bc} + \nabla^c\nabla_b\delta g_{ac} - \nabla_a\nabla_b g^{cd}\delta g_{cd} - \square\delta g_{ab}\right). \quad (\text{B.11})$$

Now it is easy to show that

$$\begin{aligned} \delta\left(\sqrt{-g}\mathcal{R}^{ab}\mathcal{R}_{ab}\right) &= \sqrt{-g}\left[-\frac{1}{2}g_{ab}\mathcal{R}_{cd}\mathcal{R}^{cd} + 2\mathcal{R}_a^c\mathcal{R}_{cb} - \mathcal{R}_a^c\nabla_b\nabla_c\right. \\ &\quad \left.-\mathcal{R}_b^c\nabla_a\nabla_c + \mathcal{R}^{cd}\nabla_c\nabla_d g_{ab} + \mathcal{R}_{ab}\square\right]\delta g^{ab}. \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} \delta I &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[f'(\mathcal{R})\mathcal{R}_{ab} - \frac{1}{2}g_{ab}f(\mathcal{R}) - \frac{b}{2}g_{ab}\mathcal{R}_{cd}\mathcal{R}^{cd} + 2b\mathcal{R}_a^c\mathcal{R}_{cb}\right. \\ &\quad \left.+ f'(\mathcal{R})g_{ab}\square - f'(\mathcal{R})\nabla_a\nabla_b - b\mathcal{R}_a^c\nabla_b\nabla_c - b\mathcal{R}_b^c\nabla_a\nabla_c + b\mathcal{R}^{cd}\nabla_c\nabla_d g_{ab} + b\mathcal{R}_{ab}\square\right]\delta g^{ab}. \end{aligned} \quad (\text{B.13})$$

Doing integration by part on the last six terms two times leads to

$$\begin{aligned} \delta I &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[f'(\mathcal{R})\mathcal{R}_{ab} - \frac{1}{2}g_{ab}f(\mathcal{R}) - \frac{b}{2}g_{ab}\mathcal{R}_{cd}\mathcal{R}^{cd} + 2b\mathcal{R}_a^c\mathcal{R}_{cb}\right. \\ &\quad \left.+ g_{ab}\square f'(\mathcal{R}) - \nabla_a\nabla_b f'(\mathcal{R}) - b\nabla_c\nabla_b\mathcal{R}_a^c - b\nabla_c\nabla_a\mathcal{R}_b^c + b\nabla_c\nabla_d\mathcal{R}^{cd}g_{ab} + b\square\mathcal{R}_{ab}\right]\delta g^{ab}. \end{aligned} \quad (\text{B.14})$$

Noting that action remains invariant under variation of the metric and putting $\delta I = 0$, results in Eq.(2.2).

C Details of derivation of Eq.(2.7)

Now $\mathcal{H}_{ab}^{(0)}$ can be written as

$$\begin{aligned} \mathcal{H}_{ab}^{(0)} &= \tilde{R}_{ab} + \delta R_{ab} - \frac{1}{2}(\tilde{R} + \delta R)(\tilde{g}_{ab} + h_{ab}) + \Lambda(\tilde{g}_{ab} + h_{ab}) \\ &= \tilde{H}_{ab}^{(0)} + \delta R_{ab} - \frac{1}{2}(\tilde{R}h_{ab} + \tilde{g}_{ab}\delta R) + \Lambda h_{ab} + \text{nonlinear terms}. \end{aligned} \quad (\text{C.1})$$

Using Eqs.(A.3), (A.5) and (A.6) in (C.1) leads to the expression presented in Eq.(2.7).

D Details of derivation of Eq.(2.9)

Now, $\mathcal{H}_{ab}^{(1)}$ can be written as

$$\begin{aligned} \mathcal{H}_{ab}^{(1)} + 2(\tilde{R} + \delta R)(\tilde{R}_{ab} + \delta R_{ab}) - 2(\nabla_a \nabla_b \tilde{R} + \delta \nabla_a \nabla_b R) \\ - \frac{1}{2}(\tilde{g}_{ab} + h_{ab}) \left[(\tilde{R}^2 + 2\tilde{R}\delta R) - 4(\square \tilde{R} + \delta \square R) \right], \end{aligned} \quad (\text{D.1})$$

which may be written as can be written as

$$\mathcal{H}_{ab}^{(1)} = \tilde{H}_{ab}^{(1)} - 2 \left(\delta \nabla_a \nabla_b R - \tilde{R} \delta R_{ab} - \tilde{R}_{ab} \delta R \right) - \tilde{g}_{ab} (\tilde{R} \delta R - 2\delta \square R) - \frac{1}{2} h_{ab} \tilde{R}^2 + \text{nonlinear terms.}$$

Making use of Eqs. (A.5), (A.6), (A.9) and (A.10) we have $\mathcal{H}_{ab}^{(1)} = \tilde{H}_{ab}^{(1)} + H_{ab}^{(1)}$, and $H_{ab}^{(1)}$ is the same as given in Eq.(2.9).

E Details of derivation of Eq.(2.11)

Eq.(2.4) can be written in the following form

$$\begin{aligned} \mathcal{H}_{ab}^{(2)} = & \left(\square \tilde{R}_{ab} + \delta \square R_{ab} \right) - \left(\nabla_c \nabla_a \tilde{R}_b^c + \delta \nabla_c \nabla_a R_b^c \right) - \left(\nabla_c \nabla_b \tilde{R}_a^c + \delta \nabla_c \nabla_b R_a^c \right) \\ & - \frac{1}{2}(\tilde{g}_{ab} + h_{ab}) \left[(\tilde{R}^{cd} + \delta R^{cd})(\tilde{R}_{cd} + \delta R_{cd}) - 2(\nabla_c \nabla_d \tilde{R}^{cd} + \delta \nabla_c \nabla_d R^{cd}) \right] \\ & + 2 \left(\tilde{R}_a^c + \delta R_a^c \right) \left(\tilde{R}_{cb} + \delta R_{cb} \right). \end{aligned} \quad (\text{E.1})$$

It can be written as

$$\begin{aligned} \mathcal{H}_{ab}^{(2)} = & \tilde{H}_{ab}^{(2)} + \delta \square R_{ab} - \delta \nabla_c \nabla_a R_b^c - \delta \nabla_c \nabla_b R_a^c + 2(\tilde{R}_a^c \delta R_{cb} + \tilde{R}_{cb} \delta R_a^c) \\ & - \frac{1}{2} \tilde{g}_{ab} (\tilde{R}^{cd} \delta R_{cd} + \tilde{R}_{cd} \delta R^{cd} - 2\delta \nabla_c \nabla_d R^{cd}) - \frac{1}{2} h_{ab} \tilde{R}^{cd} \tilde{R}_{cd} + \text{nonlinear terms.} \end{aligned} \quad (\text{E.2})$$

Now using the relations given in appendix-A we can show that

$$\tilde{R}_a^c \delta R_{cb} = \frac{3}{2} H^2 \left(\nabla_a \nabla_c h_b^c + \nabla_b \nabla_c h_a^c + 8H^2 h_{ab} - 2H^2 h' \tilde{g}_{ab} - \square h_{ab} - \nabla_a \nabla_b h' \right), \quad (\text{E.3})$$

$$\tilde{R}_{cb} \delta R_a^c = \frac{3}{2} H^2 \left(\nabla_a \nabla_c h_b^c + \nabla_b \nabla_c h_a^c + 2H^2 h_{ab} - 2H^2 h' \tilde{g}_{ab} - \square h_{ab} - \nabla_a \nabla_b h' \right), \quad (\text{E.4})$$

$$\delta \nabla_c \nabla_d R^{cd} = \nabla_c \nabla_d (\nabla^c \nabla_f h^{fd} + \nabla^d \nabla_f h^{fc}) + 2H^2 \nabla_c \nabla_d h^{cd} - 2H^2 \square h' - \nabla_c \nabla_d \square h^{cd} - \nabla_c \square \nabla^c h', \quad (\text{E.5})$$

$$\tilde{R}^{cd} \delta R_{cd} = 3H^2 \left(\nabla_c \nabla_d h^{cd} - \square h' \right), \quad (\text{E.6})$$

$$\tilde{R}_{cd} \delta R^{cd} = 3H^2 \left(\nabla_c \nabla_d h^{cd} - \square h' - 6H^2 h' \right). \quad (\text{E.7})$$

Substituting in Eq.(E.2) we obtain $\mathcal{H}_{ab}^{(2)} = \tilde{H}_{ab}^{(2)} + H_{ab}^{(2)}$ and $H_{ab}^{(2)}$ is the statement given in Eq.(2.11).

In derivation steps, the following identities have been used

$$\begin{aligned} \nabla_c \nabla_a \square h_b^c &= \square \nabla_a \nabla_c h_b^c + 2H^2 \nabla_a \nabla_c h_b^c - 2H^2 \nabla_b \nabla_c h_a^c + 2H^2 \tilde{g}_{ab} \nabla_c \nabla_d h^{cd} \\ &\quad - 2H^2 \nabla_a \nabla_b h' + 4H^2 \square h_{ab} - H^2 \tilde{g}_{ab} \square h', \end{aligned} \quad (\text{E.8})$$

$$\nabla_c \nabla_a \nabla^c \nabla_d h_b^d = \square \nabla_a \nabla_d h_b^d - H^2 \nabla_b \nabla_d h_c^d + H^2 \tilde{g}_{ab} \nabla_c \nabla_d h^{cd}, \quad (\text{E.9})$$

$$\nabla_c \nabla_a \nabla_b \nabla^c h' = \square \nabla_a \nabla_b h' - H^2 \nabla_a \nabla_b h' + H^2 \tilde{g}_{ab} \square h', \quad (\text{E.10})$$

$$\nabla_c \nabla_a \nabla_b \nabla_d h^{cd} = \nabla_a \nabla_b \nabla_c \nabla_d h^{cd} + 4H^2 \nabla_a \nabla_c h_b^c + 3H^2 \nabla_b \nabla_c h_a^c - H^2 \tilde{g}_{ab} \nabla_c \nabla_d h^{cd}, \quad (\text{E.11})$$

$$\nabla_c \nabla_a \nabla_b \nabla_d h^{cd} = \nabla_a \nabla_b \nabla_c \nabla_d h^{cd} + 4H^2 \nabla_a \nabla_c h_b^c + 3H^2 \nabla_b \nabla_c h_a^c - H^2 \tilde{g}_{ab} \nabla_c \nabla_d h^{cd}, \quad (\text{E.12})$$

$$\square \nabla_a \nabla_b h' = \nabla_a \nabla_b \square h' + 8H^2 \nabla_a \nabla_b h' - 2H^2 \tilde{g}_{ab} \square h'. \quad (\text{E.13})$$

F Details of derivation of Eq.(3.3)

By imposing the tensor field

$$\mathcal{K} = \theta \phi_1 + \mathcal{S} \bar{Z}_1 K + D_2 K_g, \quad (\text{F.1})$$

to obey the field equation

$$(Q_2^{(1)} + 4)(Q_2^{(1)} + 6)\mathcal{K} = 0, \quad (\text{F.2})$$

and making use of the following identities [20],

$$Q_2^{(1)}(\theta \phi) = \theta Q_0^{(1)} \phi, \quad (\text{F.3})$$

$$Q_2^{(1)} D_2 K_g = D_2 Q_1^{(1)} K_g, \quad (\text{F.4})$$

$$Q_2^{(1)} \mathcal{S}(\bar{Z} K) = \mathcal{S} [\bar{Z}(Q_1^{(1)} - 4)K] - 2H^2 D_2(x.Z)K + 4\theta Z.K, \quad (\text{F.5})$$

we have

$$\begin{aligned} &(Q_2^{(1)} + 4) \left([(Q_0^{(1)} + 6)\phi_1 + 4Z_1.K] \theta \right. \\ &\quad \left. + \mathcal{S} [\bar{Z}_1(Q_1^{(1)} + 2)K] + D_2 [(Q_1^{(1)} + 6)K_g - 2H^2(x.Z_1)K] \right) = 0. \end{aligned} \quad (\text{F.6})$$

Making use of Eqs.(F.3)-(F.5) in Eq.(F.6) once again, we obtain

$$\begin{aligned} &\theta [(Q_0^{(1)} + 4)(Q_0^{(1)} + 6)\phi_1 + 4Q_0^{(1)} Z_1.K + 4(Q_1^{(1)} + 2)x.Z_1 K + 16Z_1.K] + \mathcal{S} \bar{Z}_1 [Q_1^{(1)}(Q_1^{(1)} + 2)K] \\ &\quad + D_2 [(Q_1^{(1)} + 4)(Q_1^{(1)} + 6)K_g - 2H^2(x.Z_1)(Q_1^{(1)} + 6)K - 2H^2 Q_1^{(1)}(x.Z_1 K)] = 0. \end{aligned} \quad (\text{F.7})$$

Using the conditions $x.K = 0 = \partial.K$ in Eq.(2.29) we have

$$(Q_1^{(1)} + 2)K = Q_0^{(1)} K, \quad (\text{F.8})$$

from which we can write

$$Q_0^{(1)}(Z_1.K) = (Q_1^{(1)} + 2)Z_1.K. \quad (\text{F.9})$$

Substituting (F.9) in Eq.(F.7) results in

$$\theta [(Q_0^{(1)} + 4)(Q_0^{(1)} + 6)\phi_1 + 8Q_0^{(1)} Z_1.K + 16Z_1.K] + \mathcal{S} \bar{Z}_1 [Q_1^{(1)}(Q_1^{(1)} + 2)K]$$

$$+D_2 \left[(Q_1^{(1)} + 4)(Q_1^{(1)} + 6)K_g - 2H^2(x.Z_1)(Q_1^{(1)} + 6)K - 2H^2Q_1^{(1)}(x.Z_1K) \right] = 0. \quad (\text{F.10})$$

It is easy to show that

$$Q_1^{(1)}(x.Z_1K) = x.Z_1(Q_1^{(1)} - 4)K - 2Z_1.D_1K + 2xZ_1.K. \quad (\text{F.11})$$

Now combining Eqs.(F.10) and (F.11) leads to the following equation

$$\begin{aligned} & \theta \left[(Q_0^{(1)} + 4)(Q_0^{(1)} + 6)\phi_1 + 8(Q_0^{(1)} + 2)Z_1.K \right] + \mathcal{S}\bar{Z}_1 \left[Q_1^{(1)}(Q_1^{(1)} + 2)K \right] \\ & + D_2 \left[(Q_1^{(1)} + 4)(Q_1^{(1)} + 6)K_g - 4H^2 \left((Q_1^{(1)} + 5)(x.Z_1K) + Z_1.D_1K - xZ_1.K \right) \right] = 0, \end{aligned} \quad (\text{F.12})$$

which results in Eq.(3.3).

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